VII. *An Experimental Investigation into the Form of the Wave Surface of Quartz.*

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I.—INTRODUCTORY REMARKS.

ABOUT two years ago I read a paper before the Cambridge Philosophical Society describing some measurements of the "dark rings" in quartz. The present paper contains an account of similar measurements made with greatly improved apparatus, and extending over a much larger field. These "dark rings" supply a delicate method of determining the retardation of the extraordinary wave behind the ordinary in the crystal and consequently the separation between the two sheets at various points of the wave-surface.

In quartz the wave-surface may be nearly represented by a prolate spheroid surrounded by a sphere passing through the extremities of its axis. The spheroid is slightly flattened at the extremity of its axis and the sphere slightly bulged, so that the two no longer touch. The distance between the two at the extremity of the axis we know very accurately from observations on the rotatory power, while the two radii at the equator of the wave-surface are known from observations with the spectrometer.

In all the theories on the subject these three constants are assumed; so the theoretical surfaces are made to coincide with the true one in the equatorial section and to give the true distance between the two sheets at the extremity of the axis. At intermediate points their correspondence with the true surface lies open to the test of experiment. My former observations were confined to the regions near the axis, as it is only there that the bulging and flattening are at all conspicuous. But the increased facilities of a new arrangement for measuring the rings led me to extend my observations over the whole surface, so that I have now determined the gradually increasing distance between the two sheets from the axis to the equator.

This was the more necessary because the known values of the radii of the equatorial section, or, in other words, of the principal wave-velocities, are not sufficiently accurate to give a good measure of their difference. And it is upon this difference that the theoretical distance between the two sheets, even in the region near the axis, to a great extent depends.

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It will be well to give at this point some idea of the relative magnitudes involved. The difference between the equatorial radii is about $\frac{1}{200}$th part and the distance between the sheets at the extremity of the axis $\frac{1}{20000}$th part of the radius of the sphere. My determination of the distance between the sheets should not at any rate be in error by more than one part in 500 at 4° from the axis, one part in 1000 at 12°, and one part in 2000 at a considerable distance from the axis. Of that, though by no means a complete investigation of the form of the wave-surface, these observations must be admitted to be a severe test. For if one wave-velocity were to be changed while the other remained the same, the change would be indicated even though it should only amount to $\frac{1}{4000000}$th part of the whole velocity. Indeed near the axis $\frac{1}{8000000}$th part would be sensible.

The theory of the formation of the rings requires some consideration. Take a wave-front of the plane polarised incident light. By the refraction at the first surface it is resolved into two wave-fronts which are polarised in a manner dependent on the constitution of the crystal, and which travel with different velocities. Each of these on refraction at the second surface gives rise to a wave-front polarised in a manner somewhat different from its own. Let the emergent wave-front due to the extraordinary wave be B and the other A. Then B and A are parallel. If B be in advance of A by an integral number of wave-lengths, A will be coincident with a wave-front exactly similar to B, and in the same phase. But we cannot say that, therefore, the combination of these two will be exactly similar to or even polarised in the same manner as the incident wave-front, for there has been loss of light by reflection at both surfaces, and this has affected A and B differently.

The consideration of the refractive effects is greatly simplified in our case by the circumstance that the optic axis lies in the plane of incidence. Thus the principal axes of the ellipses of vibration of the waves in the quartz lie respectively in, and perpendicular to, the plane of incidence.

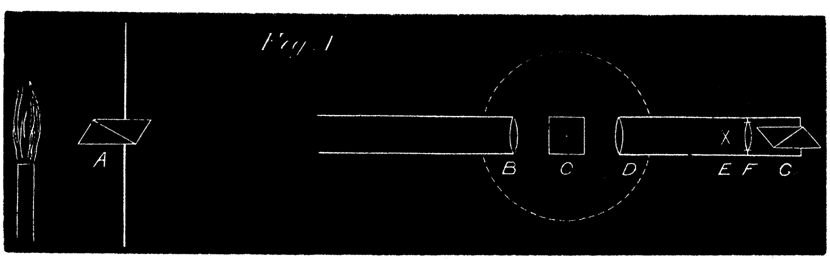
We will assume at present that the action at the surface produces no retardation of phase. Let us first suppose that the incident light is plane polarised and the vibration lies in the plane of incidence. This linear vibration on entering the crystal gives rise to two elliptic vibrations; and we may, I think, safely assume that, if the relative retardation amount to an integral number of wave-lengths these two elliptic vibrations will give rise on emergence to a single linear vibration also in the plane of incidence. For we may consider the components of these two elliptic vibrations perpendicular to the plane of incidence as due to two equal and opposite linear vibrations of the incident light. These two linear vibrations would suffer equal reductions at the first refraction, and equal reductions at the second refraction, so after both refractions they would again neutralise each other. It is indeed probable that the difference in the angle of refraction of the two waves would affect the amount of light lost by reflection, but it would be to a negligibly small extent. Thus the emergent light will be sensibly plane polarised.

Similarly, if the vibration of the incident light be perpendicular to the plane of incidence, the emergent light will be also plane polarised. Consequently, if the incident light be plane polarised in any plane, the emergent light will also be plane polarised, though not necessarily in a parallel plane. Thus, for angles of incidence giving a relative retardation of an integral number of wave-lengths, the refractive effects are equivalent to a small rotation of the plane of polarisation of the incident light. And the same is no doubt very approximately true for neighbouring angles of incidence. This small rotation can, as will appear later on, be eliminated by taking the mean of two sets of observations.

Up to this point, and indeed throughout, we assume that there is no perceptible retardation of phase at the surface. Careful observations have been made on glass with regard to this point, and the only trace of retardation found was at very high angles of incidence, and even that was highly doubtful. There seems to be no reason to expect such an effect in crystalline media more than in isotropic, at any rate when the rotatory effects are small. In the present case too the rings are very close together at the high angles of incidence, so that only a small error would result; *e.g.*, at an incidence of 75°, which is the highest incidence I have reached, the error in the diameter of a ring produced by a retardation of $\frac{\lambda}{100}$ would be only $\frac{1}{2}'$. As to any surface retardation connected with the rotatory power, we know from measurements of the rotation in plates of different thickness that it is, at any rate, very small when the light is incident normally, and passes along the axis. These measurements have been made with great accuracy, and no such effect has been even suspected. So we have a maximum limit for it, from which I deduce that it could only make itself felt slightly in the first ring, in the others not at all.

II.—The Apparatus and Measurements.

My previous observations were made with one of GROTH's polariscopes somewhat modified, as described in the paper, but since then I have found it a great improvement to use instead an ordinary spectrometer. With some slight additions the spectrometer became a most efficient instrument for measuring the rings.



The general arrangement is indicated in fig. 1. The source of light is a BUNSEN flame in the edge of which is placed a bead of sodium carbonate. The light is polarised by the NICOL A and screens are placed so that no other light can reach the eye. The slit of the collimator has been removed so the light falls on the whole breadth of the object glass B of the collimator. Here it is transformed into a convergent beam which passes through the quartz plate C. Each parallel pencil emerging from the quartz is brought to a focus in the focal plane of the object-glass D. So the needle-point E has to be carefully adjusted to lie in this plane. In this particular the spectrometer has a great advantage over the polariscope. The focal length is much greater and so the adjustment can be made with greater accuracy, and the spherical aberration, which in the polariscope is very troublesome, is so small as to produce no sensible error. F is the eye-piece and G the second NICOL by which the light is analysed.

I found that the ordinary eye-piece magnified too strongly, so I substituted a single lens and fitted it up with the NICOL G in a large cork, so that the combination could be readily exchanged for the ordinary eye-piece without disturbing the needle-point. Even with the low power the apparent size of the rings was vastly greater than with the polariscope. This was mainly due to the greater focal length D E, which was about nine inches. Though the definition of the rings suffered with the increased magnification, there was on the whole a decided gain in accuracy, and the observations became far less trying to the eyes. The spectrometer table too was furnished with a slow motion screw which was no small advantage. The spectrometer was fitted with a circle graduated on silver, and both the telescope and table were provided with verniers reading to half minutes. By replacing the slit and the ordinary eye-piece of the telescope the spectrometer could return to its ordinary use. The normal to the faces of the quartz could thus be set at right angles to the axis of rotation of telescope and table and its position determined relatively to the axis of collimation of the telescope for any given readings of the telescope and table verniers. For levelling purposes the quartz was mounted on a tripod stand resting on three adjustable screws whose rounded ends rested in a conical hole, in a V-shaped slot, and on a smooth surface, respectively on the table of the spectrometer. Each screw carried a loose nut which, when the screw had been properly adjusted, could be screwed down tight onto the tripod. This made the three screws a perfectly firm support and the tripod could be removed from the table and replaced accurately in its original position—a great convenience.

The distance A B from the polarising NICOL to the object-glass of the collimator was over four feet, so the light which fell on the lens B had passed through the NICOL nearly as a parallel pencil, and therefore was polarised approximately all in one plane. The two NICOLS could therefore be crossed with greater accuracy. In order to cross the NICOLS the quartz was removed and the cork turned till the light was quenched. In my earlier observations I simply used the sodium light for this purpose, but I found that this led to errors of two or three minutes in the smaller rings, so the crossing for all the observations given below was performed with a gas-flame placed edgeways behind the polariser. This gave ample accuracy.

[*Added May* 31, 1886.—Probably the lenses had some slight doubly refracting power, so that each turned the plane of polarisation through a small angle depending on its azimuth. In the present case however this effect was a matter of indifference, as I was measuring the angles of incidence, for which the emergent light from the quartz was polarised similarly to the incident light. In the larger rings, it is true, the quartz produced a considerable lateral displacement of the light, so it no longer passed centrically through the two object glasses. But in the larger rings the effect of any small error of crossing might, as will be seen shortly, be completely neglected.]

The plane of polarisation was inclined at 45° to the plane of incidence. If we look at the appearance presented in the polariscope when the two NICOLS are crossed, we see that all the rings are circular, and though the inner rings are tolerably uniform, the outer rings are crossed by dark brushes in directions parallel and perpendicular to the plane of polarisation. These dark brushes have the effect of making the rings very ill-defined, so that on that account alone it would be impossible to get a satisfactory measure of the 15th ring if the plane of polarisation were parallel to the plane of incidence. If now we rotate the analyser slightly we see the rings dilate, but not uniformly. The dilatation is much more rapid in the direction of the brushes. The rings cease to be circular and become four-cornered. This phenomenon was observed and explained by Sir GEORGE AIRY (Camb. Phil. Trans., vol. 4, p. 85).

It is an easy deduction from his formula for the intensity at any point of the field that the value of R for the darkest part of the ring when the NICOLS are not accurately crossed is given by

$$\sin \frac{2 R\pi}{\lambda} = \frac{2 \sin \delta}{\sin 2\gamma}$$

in the direction of the brushes, but by

$$\sin \frac{2 R\pi}{\lambda} = 2 \sin \delta \sin 2\gamma$$

in directions at 45° to the brushes. Where R is the relative retardation, δ is the small angular error of crossing, and $\tan \gamma$ is the ratio of the axes of the ellipse of vibration of either wave, $\tan \gamma$ being less than unity. The following figures will give some idea of the values of γ. For sodium light when—

$$\begin{array}{ll} \phi = 1^\circ & \gamma = 43^\circ\ 53' \\ \phi = 5^\circ & \gamma = 23^\circ\ 0' \\ \phi = 10^\circ & \gamma = 7^\circ\ 21' \\ \phi = 20^\circ & \gamma = 1^\circ\ 55' \end{array}$$

ϕ being the angle between the wave normal and the optic axis.

[When $\delta=0$, $R=n\lambda$, n being an integer; as is obvious *à priori*. Now the change in the relative retardation necessary to produce a ring gives a fair indication of the change in the radius of the ring.*] So for the larger rings, while the displacement of the ring in the 45° directions, due to a small error in crossing, is almost nil, the displacement in the parallel and perpendicular directions is extremely large. The latter, however, is masked by the general obscurity of the brushes. From this it is clear that the best position for the plane of polarisation is at an inclination of 45° to the plane of incidence.

It occurred to me, as an objection that might be of importance, that there would be some rotation of the plane of polarisation through refraction at the surface of the quartz and the surfaces of the lenses, and that putting the plane of polarisation in the 45° position made this rotation as large as possible. But it is easy to see that if the polariser and analyser be turned into the other 45° position the rotation will be in the opposite direction. Thus in one case the rings will be slightly too large, in the other just as much too small, and by taking the mean of the two the error will be eliminated. I have, therefore, taken observations throughout in these two positions of the polariser, though no effect of the kind has made itself manifest.

The first set of observations recorded below were made on a piece of quartz, which I will call Plate 1. It has four polished faces, and the section perpendicular to these faces has the figure of an equilateral triangle described on a square. The two sides of the square adjacent to the triangle were polished faces forming a parallel-sided plate. The two other faces were convenient as supplying a definite plane of reference passing through the normal to the two plate faces, though they were originally cut for quite a different purpose. Each face was about an inch square. The whole piece was perfectly clear, and the only flaw was a slight intrusion of contrary quartz into one corner. The plate faces were very nearly parallel planes, the error of planitude not exceeding 1′, and the error of parallelism not exceeding 2′. Measurements of the rings were made in two different planes. The plane which is perpendicular to all four faces I will call plane A. The plane at right angles to plane A I will call plane B. The four polished faces were very nearly in the same zone.

The observations in plane A were made in the following order. The quartz was fixed on the tripod and all four faces set parallel to the axis of rotation of the spectrometer in the usual way by means of reflections from the faces. Readings were also taken from which I could find the position of the normal to one of the plate faces relative to the axis of the telescope for any given readings of the telescope and table verniers. The tripod and quartz were then lifted off the table, the slit removed from the collimator, the ordinary eye-piece of the telescope replaced by the cork containing the low power eye-piece and analyser, the gas-flame placed behind the polariser, and the NICOLS crossed. The tripod having been returned to its old position and the gas-flame exchanged for the sodium-flame every-

* Added May 31, 1886.

thing was ready for observation. The temperature was taken, usually at the beginning and end of a set of observations, from a thermometer lying on the table by the side of the spectrometer.

The appearance seen in the telescope was the needle-point standing out against an irregular patch of light crossed by one or more dark bands. The bright patch was really the aperture of the analysing NICOL, and as the telescope was moved round one or two degrees the patch moved across the needle-point. This was due to the light which reached the eye coming from a different part of the object-glass and thus passing in a different direction through the analyser.* Thus it was really equivalent to a slight rotation of the analyser round its axis. A similar effect was produced by turning the table owing to the lateral displacement of the beam of light by the quartz. To avoid the latter evil I at first always turned the telescope till the needle-point was exactly in the centre of the patch before I took a reading. This was done in most of the observations on Plate 1. But for the small rings in Plate 2 I simply clamped the telescope in the position where the NICOLS were crossed, and kept it there. By trying exaggerated cases I found the error was very small and it tended to increase the radius on one side while diminishing it on the other.

The rings of course were fixed relatively to the quartz, so that the difference of the readings of the table vernier for the two sides of the ring being on the needle-point gave the diameter of the ring in air.

For the first half dozen rings it may be shown, that the mean of these two readings should coincide, or in other words that the rings are practically circular and have a common centre, provided that the deviation of the axis from the normal do not exceed two or three degrees. This common centre may be called the axis in air, and is related to the real optic axis by the ordinary law of refraction. From this it is easy to see how the deviation of the axis from the normal was determined from the measurements of the rings. I found that the error was $1^\circ\ 45'$ measured parallel to plane A and $18'$ measured perpendicular to plane A. It is clear since the axis lay $18'$ out of plane A that instead of measuring the diameter of a ring I was only measuring a chord. The small correction for this was however easy to apply.

But for the measurements at right angles to plane A this chord would have been perhaps only half the diameter for the first ring if I had proceeded as before. It was necessary to tilt the plate forwards till the axis in air came nearly into the plane of measurement. I was at a loss for some time how to produce this large tilt on the spectrometer with any accuracy. At last I hit

* When the direction of light incident on a NICOL'S prism is inclined to the principal plane of the prism at an angle α, then, in order that the light may be completely quenched by the prism, its plane of polarisation must be inclined at an angle $\alpha/3$ nearly, to the plane through the ray most nearly parallel to the principal plane of the prism. See a paper by the author, Phil. Mag., vol. 19 (1885), p. 320.—[June 15, 1886.]

upon the following expedient. I fixed a small piece of silvered glass on to one side of the plate so as to be approximately in the same zone as the four faces and approximately perpendicular to the axis in air. I first fixed it in its place with soft wax, adjusting it with my eye at the telescope, and then secured it firmly by pouring in electrical cement behind it. After this was attached, it was only necessary to bring its normal into the plane of measurement to secure that the diametral chord was being measured. Plane B then may be defined as passing through this normal and being approximately perpendicular to plane A.

The first ring was considerably broader and less sharply defined than the others; yet in obtaining the diameters recorded in the tables below I have relied on a single reading of each side even of the first ring. For during a long course of preliminary measurements I had found, that two readings of the first ring would seldom differ more than $\frac{1}{2}'$, though I could not be confident that the true reading lay within $\frac{1}{2}'$ of either. It seemed in fact to be easier to bring the needle back to the same point in the band than to make sure of that point being the middle of the band. To obtain real accuracy I relied on varying the conditions as much as possible. Thus the four sets of measurements recorded in Table I. were made on four different days with different positions of the polariser, analyser, and plate of quartz.

I also made a number of measurements of the larger rings in Plate 1, but these, I regret to say, have proved to be useless. The formulæ, which are compact enough when the plate is cut truly perpendicular to the axis, become hopelessly complicated when the deviation of the axis has to be introduced. Since the deviation of the axis was small I tried approximations, carrying them as far as the squares of the deviations in the two directions. But the discrepancy between the results for plane A and for plane B seemed to indicate that the cubes also should have been included.

I then determined to obtain a new plate in which the deviation of the axis should be negligibly small. I selected a good piece at HILGER's and got him to cut it roughly at right angles to the axis. I then set it up on the spectrometer, determined the magnitude and direction of the error, and sent it back to HILGER to be recut. By repeating this process, getting it recut no less than four times, I at length attained my object. This plate I will call Plate 2. It was perfectly clear and free from maccling. The two polished faces were good planes about 30 mm. square, and inclined at an angle of about $1\frac{1}{2}'$. The two planes through the normal and parallel to the sides of the square I will call plane A and plane B. The error of the axis was about $6'$ parallel to plane A and about the same parallel to plane B. This I found was small enough to be negligible throughout. The thickness was about 20 mm. only.

The first ring had nearly the same diameter as in Plate 1, but it was produced by a retardation of three wave-lengths instead of four. Owing to the smaller thickness the bands were broader and further apart than before; but I do not think this impaired the accuracy of measurement except perhaps slightly in the first two rings. I had the plate cut down to this thickness in order to secure good observations at the high angles of incidence. The object glasses of the telescope and collimator were barely 25 mm. in diameter, yet I was able to obtain good readings for the 150th ring at an angle of incidence of over 74°, which would have been quite impossible with a plate 25 mm. thick.

The observations on the small rings in Plate 2 were similar to those in Plate 1, except that three readings were taken throughout for each side of the first ring and two readings for each side of the second ring. The greatest difference between the readings was $1\frac{1}{2}'$ in the first ring and $1'$ in the second. As will be seen below the first ring gave results which were at first sight rather discordant with the others. So I made a few special observations on the first ring with a very bright sodium-flame. This was obtained, without the use of oxygen, with the aid of a thin platinum wire carrying a very small bead of carbonate of sodium. The thin wire was raised to an intense heat in the hottest part of the flame, while sufficient liquid oozed down the wire from the molten bead to replace the sodium given off. The flame was nearly if not quite as bright as if oxygen had been blown in, and the definition of the first ring was decidedly improved; though traces of other colours appeared which were a little troublesome.

The measurements of the larger rings deserve a few remarks. The rings were narrow and sharply defined, so the needle-point, could be placed with great accuracy, and another important source of error was removed, for turning the polariser or analyser slightly would not appreciably displace the ring. I believe the discrepancies between different measurements that appear in the tables arose chiefly from errors of $\frac{1}{2}'$ in reading the verniers. Owing to the lateral displacement of the light when the plate was placed obliquely, the telescope had to be moved various distances up to about 1° to either side of its mean position. To make allowance for this motion the telescope vernier had to be read. Thus each diameter given depends on four vernier readings. It is not then very surprising that two diameters should occasionally differ by $1\frac{1}{2}'$; while the mean of the four diameters is probably very accurate.

The patch of light alluded to previously as forming the field of view was, as the angle of incidence increased, gradually encroached on by the edge of the object-glass of the telescope. This was avoided by moving the telescope. But soon the edges of the quartz plate began to encroach, and this was unavoidable. When the 150th ring was on the needle-point the field of view was very narrow, so the readings were not quite as accurate as for the other rings. Still I do not think the error of setting the needle-point was ever as much as $1'$. Notwithstanding the movement of the telescope the light at the high angles of incidence entered near the edge of the object-glass; so I carefully tested the telescope for spherical aberration and found that no appreciable error could arise from this cause. As a matter of curiosity I found it was possible to obtain a reading of both sides of the 163rd ring.

The above observations relate to values of ϕ ranging from 4° to 38°. For larger values I required a plate cut parallel to the axis. So I had two new faces cut on Plate 2, at right angles to the former ones and therefore nearly parallel to the axis, forming in fact a new plate, which I will call Plate 3. The two new faces were good planes and made an angle of about $2\frac{1}{2}'$ with one another, and not more than 9′ with the axis.

In the observations on Plate 3, it was necessary to set all four faces nearly parallel to the axis of rotation of the spectrometer table so as to secure that the axis should lie in the plane of measurement. The bands were shadowy and indistinct compared with those seen in Plate 2. This was only to be expected, for we really have two distinct sets of bands due to the two sodium lines. For a given value of ϕ the retardation is very nearly the same for either line, but if it amounts to 300 wave-lengths for D_1 it will be about 300·3 wave-lengths for D_2. Thus the 300th band for D_1 will divide the distance between the 300th and 301st bands for D_2 in the ratio of 3 to 7, now in the bands observed the retardation ranged from 253 wave-lengths to 312. So it was no wonder the bands were indistinct, the wonder was rather that they were measurable at all. Fortunately however a large angular error in the measurement of a band only produced a small error in the retardation deduced. To take an extreme example, in the first band an error of 10′ in the diameter only produced an error of $\frac{1}{10000}$th part in the retardation deduced therefrom. I took two sets of measurements of these bands with the analyser in one position and two sets with it in the rectangular position. The 2nd, 5th, 15th, 30th, and 50th bands were measured in all four sets, the first only in the first set, and the 60th only in the two last. On either side of the first band I took three readings, the extreme difference between two being 14′. For the second I usually took two readings, the extreme difference in one case being 9′, the next greatest being $5\frac{1}{2}'$. For the fifth in four cases I took two readings, the extreme difference being $2\frac{1}{2}'$. But, as will be seen below, these differences are quite eclipsed by those due to small differences of temperature on different days.

Indeed if it was desired to get a very accurate measure of the change of $a-b$ with temperature a most powerful method would be to take similar observations to these, using a plate of quartz about 70 mm. thick, so that the bands for the two D lines would nearly overlap, and thus narrow and sharply-defined bands would be presented in the field of view.

The Measurement of the Thickness.

The thickness of the plates was measured with a pair of callipers made by ELLIOTT BROS. These were fitted with a vernier reading to thousandths of an inch, and the graduation was so good that it was easy to divide the vernier divisions into halves or even quarters. To avoid any error arising from the jaw faces of the callipers not being strictly plane, I clamped two pins on to the jaws of the callipers with their heads opposed to one another. The direct contact of the two pins' heads was observed with a magnifying glass, and when the quartz was interposed each pin's head was brought up into contact with its image in the surface of the quartz. Care was of course taken to set the plate of quartz at right angles to the length of the callipers. This gave the thickness of Plate 1, 1·0885 inch or 27·65 mm., of Plate 2, ·7865 inch or 19·977 mm.

These are probably correct within $\frac{1}{2000}$th inch. If I had measured Plate 3 in the same way I might have got an error of $\frac{1}{1000}$th part in the ratio of its thickness to that of Plate 2. Now it was the ratio that I was anxious to get accurately, for, as will appear below, I was more interested to see whether $a-b$ would come out from my calculation a constant, than to find out what might be the value of that constant. I had arranged that Plate 3 should be of nearly the same thickness as Plate 2. If then I could get an accurate value of the difference between them, it would supply an accurate value of the ratio. This difference was measured with a spherometer reading to ·001 mm. The plate was laid on three steel points firmly fixed into a table. The three legs of the spherometer rested on a mahogany block clamped to the table. Each leg had been pressed gently into the wood to make a small pit, and I found that the spherometer could be lifted off and replaced several times, and yet give the same reading. I found the difference of thickness to be ·275 mm. within ·002 mm. Thus I obtained the ratio correct to $\frac{1}{10000}$th part.

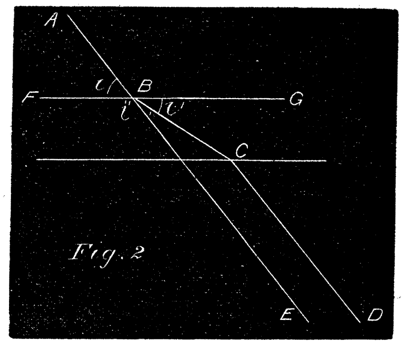
III.—Calculation and Discussion of Results.

It is convenient to discuss the observations on small values of ϕ separately from the others. Near the optic axis the additional separation of the sheets of the wave-surface, due to the rotatory property, has to be treated as of magnitude comparable with the separation given by Huyghens' construction, both being very small. While at a little distance from the axis the Huyghenian separation is greatly increased, and the additional portion sinks to the magnitude of a small correction. Thus for each case a special formula of approximation is appropriate. The boundary may be drawn at $\phi=12°$, though either formula will do for some distance on either side of this boundary.

The Small Rings.

In my former paper I calculated the radii of the various small rings on MacCullagh's theory and compared their values with observation. In order to compare the observations with other theories the results will now be presented in a different form, and it is convenient to introduce a quantity D to denote the number of wave-lengths by which one wave lags behind the other in air after the light has traversed normally a plate of quartz 1 mm. thick, the normal to whose faces makes an angle ϕ with the optic axis. Let R denote the retardation actually observed when the light passes obliquely through the plate. We have to determine the relation between R and D. If we were free to assume that the two wave-fronts in the crystal might be treated as parallel this would be very simple, and in my former paper I made this assumption; but further consideration has convinced me that the assumption is inadmissible, at any rate without a good deal more explanation. It is true that the two wave-fronts are very nearly parallel, but it is also true that the retardation to be measured depends on the small difference between two nearly equal velocities. I therefore give the following more rigorous investigation, which leads, as will be seen, to the same result as the questionable assumption.

Fig. 2.



Let FGC in fig. 2 represent a plate of quartz. Let ABCD be a wave-front; of which the part AB has not reached the quartz, and the part BC is an "ordinary" wave-front in the crystal. Let ι be the angle of incidence, ι' ι'' the two angles of refraction.

Then the retardation of the ordinary wave in passing through the plate is the perpendicular distance between CD and ABE

$$=\mathrm{T}\frac{\sin(\iota-\iota')}{\sin\iota'}=\mathrm{T}(\sin\iota\cot\iota'-\cos\iota).$$

Similarly the retardation of the extraordinary wave is

$$\mathrm{T}(\sin\iota\cot\iota''-\cos\iota).$$

So the relative retardation is

$$\mathrm{T}\sin\iota\,(\cot\iota''-\cot\iota') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is a well-known result, the above proof being due to Dr. ROUTH. I have inserted it for the sake of completeness, and to show that so far no approximations have been made.

Let s' s'' be the two wave-velocities in the quartz, then

$$\left.\begin{array}{l}\sin \iota' = s' \sin \iota \\ \sin \iota'' = s'' \sin \iota\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let

$$s'' = s' - \sigma, \qquad \iota'' = \iota' - \epsilon.$$

We shall see that σ/s' is always less than $\frac{1}{1500}$, so that although R contains σ as a factor the squares of σ/s' may be neglected, while ϵ/ι' is smaller still.

From (2) we obtain

$$\epsilon \cos \iota' = \sigma \sin \iota.$$

Then

$$\cot \iota'' - \cot \iota' = \frac{\epsilon}{\sin^2 \iota'} = \frac{\epsilon}{s'^2 \sin^2 \iota} = \frac{\sigma}{s'^2 \sin \iota \cos \iota'}$$

Whence

$$R = T \sin \iota \, (\cot \iota'' - \cot \iota') = \frac{T\sigma}{s'^2 \cos \iota'}$$

Again

$$D\lambda = \frac{1}{s''} - \frac{1}{s'} = \frac{\sigma}{s'^2} = \frac{R \cos \iota'}{T}$$

I have examined the effect of retaining the squares of σ and ϵ in finding this relation, and find that the above equation is true within one part in ten thousand.

For the successive rings we have $R = n\lambda$, where n is an integer.

So

$$D = \frac{n \cos \iota'}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where T is measured in millimetres.

All the nine theoretical formulæ I have examined for the relative retardation near the axis in quartz take one or other of the following forms :—

$$\left.\begin{array}{l} D^2 = P_1^2 \sin^4 \phi + D_0^2 \\ \text{or} \quad D^2 = P_2^2 \sin^4 \phi + D_0^2 \cos^4 \phi \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where P_1 and P_2 are constants which have certain values assigned to them.

D_0 is the value of D when $\phi = 0$, and is known from the rotation. If ρ be the rotation, measured in degrees, of a millimetre plate of quartz $\rho = 180 D_0$.

If the plate be cut strictly at right angles to the optic axis $\phi = \iota'$ and the diameter of each ring is 2ι. Using the equation $\sin \phi = a \sin \iota$, where $1/a$ is the ordinary index, and eliminating D between (3) and (4) we have

$$n^2 \cos^2 \phi = T^2 (P_1^2 a^4 \sin^4 \iota + D_0^2)$$
$$n^2 \cos^2 \phi = T^2 (P_2^2 a^4 \sin^4 \iota + D_0^2 \cos^4 \phi)$$

Thus, from the observed values of ι, we can obtain values of P_1 and P_2 for each ring. According to one set of theories P_1, according to the other P_2 should be always the same.

The observations on Plate 1 are exhibited in Table I. The first line gives the value of n; the next four the observed values of the diameters taken in plane A or in plane B with the principal plane of the analyser in the position indicated; the sixth line the result of taking the mean of the two diameters in plane A and correcting it for the deviation of the axis from the normal; the seventh the same for plane B; the eighth the mean of the two last. The ninth shows the diminution of P_1 or P_2 for an increase of 1′ in the diameter, so as to give an indication of the probable accuracy. The next line gives the approximate value of ϕ, and the last two the values of P_1 and P_2 deduced from each ring.

TABLE I. (Plate 1).

Temp.	Value of n.	4.	5.	6.	7.	8.	13.	18.
19½°	Plane A analyser \ . .	12° 49½′	16° 39′	19° 17′	21° 25½′	23° 16¾′	30° 40′	36° 27′
15¼°	" " / . .	12° 48½′	16° 37′	19° 15½′	21° 24½′	23° 17′	30° 38½′	36° 26′
13½°	Plane B " \ . .	12° 52′	16° 41′	19° 17½′	21° 25½′	23° 17½′	30° 38½′	36° 25′
14½°	" " / . .	12° 51′	16° 40½′	19° 17′	21° 24¾′	23° 17½′	30° 37½′	36° 25′
	Corrected mean plane A .	12° 50½′	16° 39′	19° 16¾′	21° 25′	23° 17′	30° 38½′	36° 25′
	" " " B .	12° 52′	16° 41¼′	19° 17½′	21° 25½′	23° 18′	30° 38½′	36° 25½′
16°	Mean corrected diameter .	12° 51¼′	16° 40′	19° 17′	21° 25¼′	23° 17½′	30° 38½′	36° 25¼′
	Change of P_1 or P_2 per 1′ in diameter	·040	·030	·026	·024	·022	·0165	·014
	Approximate ϕ	4° 9′	5° 23′	6° 14′	6° 55′	7° 31′	9° 51′	11° 40′
	" P_1	15·084	15·181	15·204	15·221	15·232	15·272	15·285
	" P_2	15·258	15·290	15·295	15·297	15·290	15·309	15·309

The observations on the small rings of Plate 2 are exhibited in Table II.

TABLE II. (Plate 2).

Temp.	Value of n.	3.	4.	5.	6.	7.	9.	13.
22°	Plane A analyser / . .	13° 38¾′	18° 10′	21° 16½′	23° 48½′	26° 5′	29° 58½′	36° 26½′
21°	" " \ . .	13° 36½′	18° 8½′	21° 15′	23° 48½′	26° 3½′	29° 56½′	36° 26½′
18°	Plane B " / . .	13° 34⅓′	18° 7¾′	21° 14½′	23° 48′	26° 3′	29° 56½′	36° 25½′
22°	" " \ . .	13° 37′	18° 10½′	21° 16′	23° 49½′	26° 3′	29° 57′	36° 26′
20¾°	Mean diameter	13° 36½′	18° 9¼′	21° 15½′	23° 48½′	26° 3½′	29° 57′	36° 26′
	Approximate ϕ	4° 24′	5° 51½′	6° 51½′	7° 40½′	8° 23½′	9° 38′	11° 41′
	" P_1	15·054	15·207	15·220	15·260	15·249	15·258	15·269
	" P_2	15·220	15·293	15·290	15·311	15·295	15·292	15·292

On looking at these tables we see that P_1 gives values steadily increasing with ϕ, while P_2 remains very nearly the same throughout. I think that the final values of the diameters of the four larger rings in each table should be correct within ½′ or ¾′, while the first three should not be in error more than 1′. In Table I. there is a line indicating the change in P_1 or P_2 corresponding to 1′ in the diameter. So we see that P_2 comes out as a constant within the limits of experimental error except for the first ring in Plate 2. It was for this reason that I made a special set of measurements of the first ring in Plate 2 with a very bright light. The values obtained in three different positions of the analyser and plate were 13° 36¾′, 13° 34¾′, and 13° 37′, mean 13° 36′, mean temperature 22½°. This gives precisely the same value, 15·220. Plate 1 also gives a decided drop in P_2 for the first ring which is barely within the limits of error above assigned. Now a change of $\frac{1}{1000}$th in the value of ρ employed in the calculation would alter P_2 by ·0275 for the first ring in Plate 2, and by ·0344 for the first ring in Plate 1. Thus a decrease of two parts in the thousand in the value of the rotation would bring the deviation of P_2 from the constant value well within the limits of error. The other rings are much less dependent on ρ; thus the same change in ρ would in the second ring only increase P_2 by ·0086. The most extended observations on the rotation are those of SORET and SARASIN ('Comptes Rendus,' tome 95 (1882), pp. 635–638, &c.). By the method of FIZEAU and FOUCAULT they obtain for D_1 21·73°, for D_2 21·69°, mean 21·71°; while by direct observation with sodium light they obtain 21·73°. I have calculated from the mean of these 21·72, applying the temperature correction which is an addition of ·00324 for each degree rise of temperature. The deduction of two parts in a thousand will bring the value down to 21·68°. Another, and, in my opinion, better explanation of the peculiar result from the first ring will be given later on.

Omitting the first ring my observations give P_2 a constant value of $15{\cdot}30 \pm {\cdot}01$. We have to examine how this agrees with the values assigned by the theories.

CAUCHY gives

$$P_2=\frac{a-b}{a^2\lambda}=15{\cdot}351$$

where a and b are the wave-velocities at right angles to the axis.

LOMMEL gives

$$P_2=\frac{1-a^2}{1-b^2}\,\frac{a+b}{2a}\,\frac{a-b}{a^2\lambda}=15{\cdot}178\,;$$

KETTLER gives

$$P_2=\frac{a+b}{2b}\,\frac{a-b}{a^2\lambda}=15{\cdot}486\,;$$

SARRAU gives

$$P_2=\frac{a+b}{2a}\,\frac{a-b}{a^2\lambda}=15{\cdot}306.$$

The discrepancy between CAUCHY'S value and the observed value is outside any probable errors of observation. It corresponds to a change of 4′ in the diameter of the 7th and 11th rings. Such an error seems out of the question. LOMMEL and

KETTLER are still further from the truth. The other five, MACCULLAGH, CLEBSCH, LANG, BOUSSINESQ, and VOIGT have the first form of expression, giving

$$P_1 = \frac{a+b}{2a}\,\frac{a-b}{a^2\lambda} = 15{\cdot}306.$$

Thus SARRAU alone succeeds in explaining the observations satisfactorily, though VOIGT does put in a proviso that his D_0 may vary with ϕ.

In obtaining the above figures the difficulty arose that $a-b$ was known only roughly from spectrometer observations. I have, therefore, had to use the value of $a-b$ deduced on SARRAU'S or MACCULLAGH'S theories from the observations on the very large values of ϕ, viz., $a-b = {\cdot}003793 \pm {\cdot}000002$.* It might perhaps be objected that it is not fair to test one theory with the value of constant obtained from another. But it should be remembered that the separation between the two sheets changes very slowly with ϕ when ϕ is nearly 90°, and that I have obtained a measurement for $\phi = 85° \, 37'$. Thus on any theory I should have obtained much the same value of $a-b$ from this measurement.

Of the above writers CAUCHY, LOMMEL, and LANG give the actual expressions I have quoted. The others only give equations to the wave-surface, from which I have had to deduce the expressions given above. MACCULLAGH, CLEBSCH, LANG, BOUSSINESQ, and VOIGT all give the same wave-surface, viz.,

$$(s^2 - a^2)(s^2 - a^2 \cos^2 \phi - b^2 \sin^2 \phi) = q^2$$

where q is a constant depending on the rotatory power. SARRAU'S wave-surface, neglecting certain constants, f_1 and g_1, which he says are probably small, is

$$(s^2 - a^2)(s^2 - a^2 \cos^2 \phi - b^2 \sin^2 \phi) = q^2 \cos^4 \phi.$$

Thus these six theories agree with the Huyghenian construction when the rotatory term is neglected. This is *à priori* a great point in their favour, since we know from STOKES'S and GLAZEBROOK'S observations that in Iceland Spar the Huyghenian construction is a very close approximation to the truth.

The more complete form of SARRAU'S wave-surface will be examined towards the end of the paper.

The references are as follows :—

MACCULLAGH, Trans. Roy. Ir. Acad., vol. 17 (1837), p. 463.

CAUCHY, Ann. de Chim. et de Phys., sér. 3, tome 30 (1850), p. 68 ; a paper by JAMIN.

* A better value is ·0037945±·0000005, which gives for SARRAU $P_2 = 15{\cdot}312 \pm {\cdot}002$. Thus, though SARRAU'S formula agrees with observation much better than any of the others, there is a small outstanding discrepancy in the three larger rings in Table II., which is as hard to account for by errors of observation, as the great drop in the first ring. This is discussed, and an explanation given near the end of the paper.—[May 31, 1886.]

CLEBSCH, 'CRELLE'S Journal,' band 57 (1859), p. 356.
SARRAU, Liouville, sér. 2, tome 13 (1868), p. 101.
BOUSSINESQ, Liouville, sér. 2, tome 3 (1868), p. 335.
LANG, Pogg. Ann. Ergbd., band 8 (1878), p. 622.
LOMMEL, Wied. Ann., band 15 (1882), p. 389.
KETTLER, Wied. Ann., band 16 (1882), p. 109.
VOIGT, Wied. Ann., band 19 (1883), p. 898.

For some account of most of these theories see GLAZEBROOK'S 'Report on Physical Optics,' Brit. Assoc., 1885.

Corrections for error of axis in Plate 1.

The calculations were made as though the optic axis were coincident with the normal to the faces. To compensate for this certain small corrections have to be applied to the observed diameters. The error parallel to plane A was 1° 45′, parallel to plane B 18′. First consider the measurements in plane A, and take the error $\alpha = 1° 45'$ by itself. It is clear that we may calculate what the error would be, on the assumption that MACCULLAGH'S theory were true, for any value of ϕ which is near the true value, and take this with sufficient accuracy to be the actual error on any one of the theories.

Let $\phi = \phi_1$ give a band on one side. The angle of refraction is then $\phi_1 - \alpha$, so we have

$$\frac{n^2 \cos^2 (\phi_1 - \alpha)}{\mathrm{T}^2} = \mathrm{P}_1^{\,2} \sin^4 \phi_1 + \mathrm{D}_0^{\,2}$$

Solving we get ϕ_1, and from the equation $\sin (\phi_1 - \alpha) = a \sin \iota_1$ we obtain ι_1, the angle of incidence on that side.

Similarly, putting α for $-\alpha$ we obtain ι_2 the angle of incidence on the other side. The observed diameter is then $\iota_1 + \iota_2$, and we take ϕ_3 to be the value of ϕ which would give the dark band in a piece correctly cut, where $\sin \phi_3 = a \sin \frac{\iota_1 + \iota_2}{2}$. But the true value is ϕ_4 where

$$\frac{n^2 \cos^2 \phi_4}{\mathrm{T}^2} = \mathrm{P}_1^{\,2} \sin^4 \phi_4 + \mathrm{D}_0^{\,2}$$

By calculating ϕ_3 and ϕ_4 we obtain the proper correction to apply to ϕ_3, and therefore to the observed diameter. In practice I found that the correction could be obtained with sufficient accuracy on the assumption that $\phi_1 = \phi_2 = \phi_4$.

On this score I had to subtract from the observed diameters of the successive rings

$\frac{1}{2}'$ $\frac{1}{2}'$ $\frac{3}{4}'$ $1'$ $1'$ $1\frac{1}{2}'$ $2'$

Again the axis lay 18′ above the plane of measurement, so a chord of the circular ring was measured instead of the full diameter. On this score the following corrections had to be added

$$2' \quad 1\tfrac{1}{2}' \quad 1\tfrac{1}{4}' \quad 1' \quad 1' \quad \tfrac{3}{4}' \quad \tfrac{1}{2}'$$

On the whole I had to add

$$1\tfrac{1}{2}' \quad 1' \quad \tfrac{1}{2}' \quad 0' \quad 0' \quad -\tfrac{3}{4}' \quad -1\tfrac{1}{2}'$$

Secondly, for the measurements in plane B.

Now $\alpha=18'$ only, so the corrections on this score are negligible. In this case the plate was tilted till the apparent position of the axis in air lay in the plane of measurement. So ι was no longer the radius of the rings in air and ι' consequently had to be altered. To allow for this I had to add to the successive diameters

$$\tfrac{1}{2}' \quad \tfrac{1}{3}' \quad \tfrac{1}{3}' \quad \tfrac{1}{3}' \quad \tfrac{1}{3}' \quad \tfrac{1}{3}' \quad \tfrac{1}{2}'$$

The corresponding corrections for Plate 2 would be infinitesimal.

The Large Rings.

Since MacCullagh's theory will be found to be in tolerable agreement with the measurements obtained for the larger values of ϕ, it is convenient to take it as a basis and discuss later what modifications would make the agreement more perfect.

MacCullagh starts by assuming the equations of motion

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2}&=A\frac{d^2\xi}{dz^2}+C\frac{d^3\eta}{dz^3} \\ \frac{d^2\eta}{dt^2}&=B\frac{d^2\eta}{dz^2}-C\frac{d^3\eta}{dz^3} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

where ξ, η are the displacements parallel to x, y at any time t, and the axis of z is the wave normal.

Also

$$\left.\begin{aligned} A&=a^2 \\ B&=a^2\cos^2\phi+b^2\sin^2\phi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

C is a constant.

From these equations we get the wave-surface

$$(s^2-A)(s^2-B)=\frac{4\pi^2C^2}{s^2\lambda^2} \quad . \quad . \quad . \quad . \quad (7)$$

which has three sheets; the third sheet, however, corresponds to an index of refraction of about 40,000; so for physical purposes we need not consider it.

Now we shall find the whole effect of C^2 is so small, not exceeding $\frac{1}{150}$th part of the chief terms in the final formula, that we may not merely reject its squares but also put $B=a^2$ in the small terms, except, of course, in the factor $B-a^2$ itself.

Resuming the rotation used for the smaller rings we have

$$\left.\begin{aligned} s'^2&=a^2+\frac{4\pi^2C^2}{a^2\lambda^2(a^2-B)} \\ s''^2&=B+\frac{4\pi^2C^2}{B\lambda^2(B-a^2)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$\sin\iota'=s'\sin\iota, \qquad \sin\iota''=s''\sin\iota.$$

By equation (1) the retardation R is

$$T\sin\iota\,(\cot\iota''-\cot\iota')$$

Assuming that the normal to the plate coincides with the optic axis we have for the ordinary wave

$$B=a^2\cos^2\iota'+b^2\sin^2\iota'$$

So $a^2-B=(a^2-b^2)\sin^2\iota'=(a^2-b^2)a^2\sin^2\iota$, neglecting C.

For the extraordinary wave

$$B=a^2\cos^2\iota''+b^2\sin^2\iota'', \qquad a^2-B=(a^2-b^2)\sin^2\iota''=(a^2-b^2)a^2\sin^2\iota,$$

with sufficient accuracy. Hence

$$\left.\begin{aligned} s'^2&=a^2+K \\ s''^2&=B-K \end{aligned}\right\}$$

where

$$K=\frac{4\pi^2C^2}{a^4\lambda^2(a^2-b^2)\sin^2\iota}.$$

To obtain the part of R not involving C we have

$$\begin{aligned}\sin^2\iota''&=B\sin^2\iota \\ &=(a^2\cos^2\iota''+b^2\sin^2\iota'')\sin^2\iota\,;\end{aligned}$$

therefore

$$\cot\iota''=\frac{\sqrt{1-b^2\sin^2\iota}}{a\sin\iota}.$$

Similarly

$$\cot\iota'=\frac{\sqrt{1-a^2\sin^2\iota}}{a\sin\iota}\,;$$

therefore

$$R=\frac{T}{a}(\sqrt{1-b^2 \sin \iota}-\sqrt{1-a^2 \sin^2 \iota})$$
$$+ \text{ a term involving C.}$$

This term

$$=T \sin \iota\left(\frac{d \cot \iota''}{dK}-\frac{d \cot \iota'}{dK}\right)K,$$

where K is to be put zero in each differential coefficient after differentiation.

$$\frac{d \cot \iota''}{dK}=-\frac{1}{\sin^2 \iota''}\frac{d\iota''}{dK}=-\frac{1}{a^2 \sin^2 \iota}\frac{d\iota''}{dK}$$

and

$$\sin^2 \iota''=(a^2-K) \sin^2 \iota$$

$$\sin 2\iota''\frac{d\iota''}{dK}=-\sin^2 \iota$$

therefore

$$\frac{d \cot \iota''}{dK}=\frac{1}{a^2 \sin 2\iota''}=\frac{1}{2a^3 \sin \iota\sqrt{1-a^2 \sin^2 \iota}}$$

Similarly

$$\frac{d \cot \iota'}{dK}=-\frac{1}{2a^3 \sin \iota\sqrt{1-a^2 \sin^2 \iota}}$$

therefore

$$R=\frac{T}{a}(\sqrt{1-b^2 \sin^2 \iota}-\sqrt{1-a^2 \sin^2 \iota})$$
$$+\frac{4\pi^2 C^2 T}{a^4\lambda^2(a^2-b^2) \sin^2 \iota}\;\frac{1}{a^3\sqrt{1-a^2 \sin^2 \iota}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This equation is not true for small values of ι or ϕ, for we have assumed in the course of the proof that K is small. We cannot, therefore, put $\phi=0$ and obtain R_0 from (9), but by putting $\phi=0$ in (7) we obtain

$$\frac{\rho}{180}=\frac{1}{s''\lambda}-\frac{1}{s'\lambda}=\frac{2\pi C}{a^4\lambda^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equation (9) is not convenient for calculation, for each of the radicles in the bracket is very large compared with their difference. And it is on their difference that R depends. Remembering that $R=n\lambda$, we may put (9) into the form

$$\frac{(a-b)(a+b) \sin^2 \iota}{\sqrt{1-b^2 \sin^2 \iota}+\sqrt{1-a^2 \sin^2 \iota}}=\frac{an\lambda}{T}-\frac{a^2\lambda^2}{a^2-b^2}\left|\frac{\rho}{180}\right|^2\frac{1}{\sin^2 \iota\sqrt{1-a^2 \sin^2 \iota}} \quad . \quad . \quad (11)$$

Of the quantities involved in this equation a, b, and λ are known with great accuracy, certainly within one part in ten thousand. T also is known with considerable accuracy, but the value of $a-b$ is uncertain, to the extent of perhaps 1 per cent. The third term is small compared with the others, and so requires only approximate values of $a-b$ and c. We may therefore use (11) as an equation to find $a-b$, inserting the observed value of ι for each of the rings measured. If the value of $a-b$ does not come out the same for each ring, then MACCULLAGH's theory is inconsistent with the facts, unless indeed the discrepancies may be attributed to errors of observations.

This mode of presenting the results of the measurements possesses the great advantage that it shows at once how far the observed wave-surface departs from the theoretical one. For instance, if for a certain value of ϕ $a-b$ comes out $\frac{1}{1000}$th part too large, this indicates that the real separation of the two sheets is very approximately $\frac{1}{1000}$th part greater than that given by the theory. This is true throughout, except for the first two small rings.

The results from the large rings in Plate 2 are exhibited in Table III. The first line gives the value of n. The next four lines give the observed diameters, two sets of observations having been taken in plane A and two in plane B. The oblique strokes indicate the position of the principal plane of the analysing NICOL. To each set is attached the mean temperature at which it was taken. The sixth line gives the mean observed diameters. The seventh the values of $a-b$ deduced from MACCULLAGH's theory. The eighth the change in $a-b$, which would be produced by an error of 1′ in the measured diameter; in this, to save space, the first five ciphers after the decimal point are omitted. The ninth line gives the value of ϕ.

TABLE III. (Plate 2).

Temp.	Value of n.	22.	32.	52.	77.	102.	127.	152.
$21\frac{1}{2}°$	Plane A analyser ╲ . .	47° 54′	58° $11\frac{1}{2}$′	75° 13′	93° 26′	110° $23\frac{1}{2}$′	127° $50\frac{1}{2}$′	148° 50′
$22\frac{3}{4}°$	" " ╱ . .	47° $53\frac{1}{2}$′	58° $11\frac{1}{2}$′	75° 14′	93° 25′	110° 23′	127° 50′	148° 51′
$22\frac{1}{2}°$	Plane B analyser ╲ . .	47° 53′	58° $10\frac{1}{2}$′	75° $12\frac{1}{2}$′	93° 25′	110° $23\frac{1}{2}$′	127° 50′	148° $52\frac{1}{2}$′
23°	" " ╱ . .	47° 53′	58° 10′	75° 13′	93° $24\frac{1}{2}$′	110° 24′	127° 50′	148° $51\frac{1}{2}$′
$22\frac{1}{2}$′	Mean diameter	47° $53\frac{1}{4}$′	58° 11′	75° 13′	93° 25′	110° $23\frac{1}{2}$′	127° $50\frac{1}{2}$′	148 $51\frac{1}{2}$′
	$a-b$ MACCULLAGH . . .	·0037884	·0037899	·0037913	·0037922	·0037931	·0037925	·0037931
	Change in $a-b$ per 1′ in diameter × 10^5 . . .	·247	·197	·142	·102	·075	·053	·033
	Approximate ϕ	15° 14′	18° 21′	23° $16\frac{1}{2}$′	28° $7\frac{1}{2}$′	32° 7′	35° 34′	38° 36′
	$a-b$ SARRAU	·0037916	·0037922	·0037927	·0037931	·0037939	·0037932	·0037936

The eighth line is an important one since the accuracy with which $a-b$ is determined varies so much. Thus, while an error of 1′ in the diameter of the 150th ring gives a change of one part in eleven thousand, the same error in the 20th ring gives a change of one part in fifteen hundred. To bring up the numbers in the seventh line to one constant value we shall have to suppose errors of 2′, $1\frac{1}{2}$′, $1\frac{1}{2}$′, and 1′ for $n=22$, 32, 52, and 77 respectively. But it is very improbable that the mean observed value of the diameter of any ring, except, perhaps, the 150th, is more than 1′ in error; so MACCULLAGH's theory must be condemned.

Now, the theory that explained the observations on the small rings most satisfactorily was that of SARRAU. His wave-surface may be obtained from MACCULLAGH's by making C vary as $\cos^2\phi$. So we must modify equation (11) by introducing the factor $\cos^4\phi$ into the third term. Calculating from the modified formula we obtain the numbers given in the tenth line of the table. The departures from constancy can now be explained by errors of less than $1'$ in the ring diameters, yet the numbers still suggest a gradual rise of $a-b$ with ϕ. We shall return to this point in discussing the results from Plate 3.

The Bands in Plate 3.

As ϕ increases the correction due to the rotatory term becomes rapidly smaller. Thus for the 150th ring in Plate 2 the change in the deduced value of $a-b$, which would be made by simply calculating from the Huyghenian construction, would be only one part in five thousand, and it may be readily shown that in Plate 3 it would never exceed one part in ten thousand. These figures refer to MACCULLAGH's theory, and in SARRAU's expressions the rotatory term is smaller still. By neglecting this term the formulæ for Plate 3 are greatly simplified.

We have now

$$s'^2=a^2$$

$$s''^2=a^2\cos^2\phi+b^2\sin^2\phi.$$

Since the plate is cut parallel to the axis we have for the extraordinary wave

$$\phi+\iota''=90^\circ;$$

therefore

$$\sin^2\iota''=(a^2\sin^2\iota''+b^2\cos^2\iota'')\sin^2\iota$$

and

$$\cot\iota''=\frac{\sqrt{1-a^2\sin^2\iota}}{b\sin\iota};$$

while

$$\cot\iota'=\frac{\sqrt{1-a^2\sin^2\iota}}{a\sin\iota}.$$

But

$$R=T\sin\iota\,(\cot\iota''-\cot\iota')\quad\text{and}\quad R=n\lambda$$

therefore

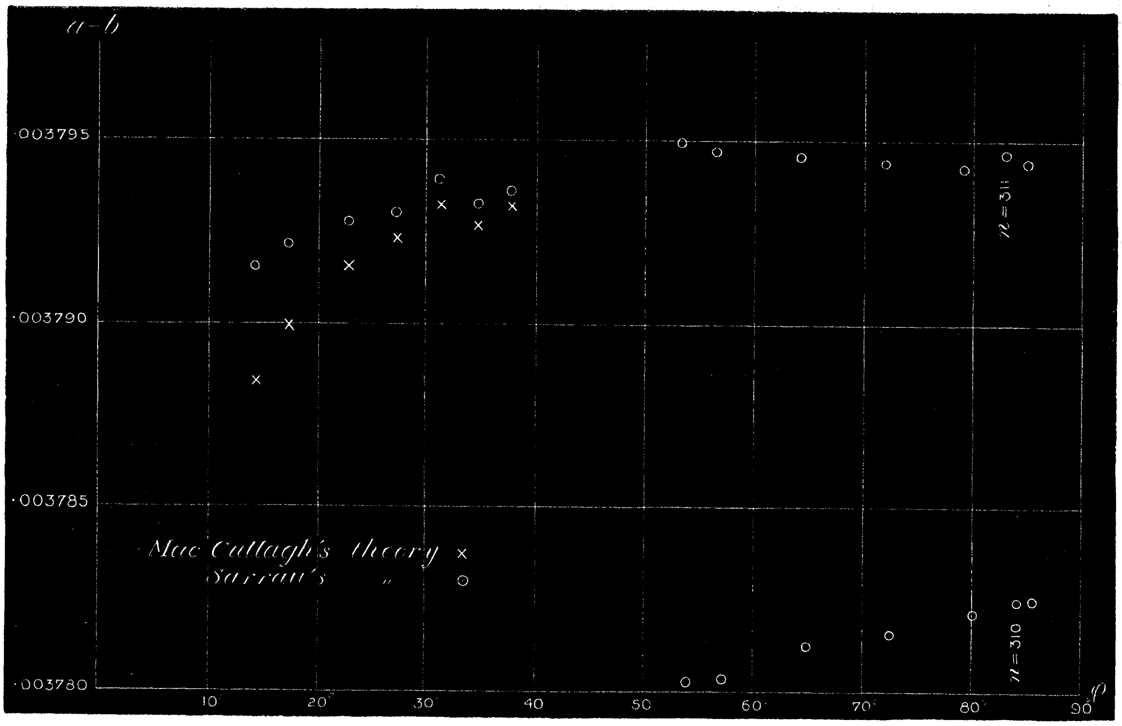
$$n\lambda=T\cos\iota'\left(\frac{1}{b}-\frac{1}{a}\right),$$

or

$$a-b=\frac{nab\lambda}{T\cos\iota'}\quad\ldots\ldots\ldots\ldots\quad(12)$$

In calculating from the observations ι was taken to be half the angular distance from any band on one side of the normal to the corresponding band on the other side. For convenience I will call this distance the diameter. Then ι' was deduced from ι by the relation $\sin \iota' = a \sin \iota$, and $a-b$ determined from the formula (12). But now the question occurs, what value must be given to n. In Plate 2 n was easily determined for the first ring, and therefore for all the rest, by a rough calculation from the known rotatory power. But in Plate 3 n is so large that tolerably accurate values of the constants are required to distinguish between one integer and the next. Indeed, without the observations on Plate 2, I should have been unable to decide between five or six different integers. However with the aid of these observations the difficulty disappears. Using the value $a-b=\cdot 0037935$ in equation (12) I obtained for the second band, reckoning from the normal, after correcting for temperature, $n=310\cdot 91$, so I concluded that the true value was 311.

Fig. 3.



Perhaps the best evidence on this point is supplied by the diagram, fig. 3. In this diagram the ordinates represent $a-b-\cdot 003780$, while the abscissæ are the values of ϕ. The small circles are got from SARRAU's theory, the crosses from MACCULLAGH's. For $\phi > 50°$ where the two theories agree there are two lines of small circles. The upper line is obtained on the supposition that $n=311$ for the second band, the lower line on the supposition that $n=310$.

The diameters obtained in four sets of measurements are given in Table IV. The first line gives the number of the band reckoned from the normal. The seventh the resulting values of $a-b$, omitting the decimal point and two ciphers, and the eighth the change in $a-b$ corresponding to 10′ in the diameter, omitting the decimal point and five ciphers. This line shows that the large discrepancies between different measurements mean only very slight variations in $a-b$. They are no doubt due mainly to changes of temperature. The temperature of course varied one or two degrees during each set of observations, and no doubt the temperature of the quartz was slightly different from that of the thermometer. The ninth line gives the values of ϕ, and the tenth the values of $a-b$ corrected to the temperature $22\frac{1}{2}°$. This was the mean temperature of the measurements of the large rings in Plate 2. The correction to $a-b$ per degree of temperature is $-·000000415$. The last line gives the values of $a-b$, taken on the supposition that $n=310$ for the second band and corrected for temperature. We notice that the greatest difference between the numbers in the tenth line is ·0000006, being about one part in six thousand. This is a strong confirmation of the truth of the Huyghenian construction for the region far removed from the axis, extending as it does from $\phi=54°$ to $\phi=85°$. The mean is $a-b=·0037945$.

TABLE IV. (Plate 3).

Temp.	No. of Band.	1.	2.	5.	15.	30.	50.	60.
20°	Analyser ╱	13° 34′	19° 35′	31° 28′	56° $10\frac{1}{2}$′	82° $25\frac{1}{2}$′	113° 37′	..
20°	„ 〈	..	19° $50\frac{1}{2}$′	31° 42′	56° 14′	82° 27′	113° $35\frac{1}{2}$′	..
22°	„ ╲	..	19° 13′	31° 25′	56° $5\frac{1}{2}$′	82° $20\frac{1}{2}$′	113° 29′	130° $35\frac{1}{2}$′
21°	„ ╲	..	19° 28′	31° 27′	56° 6′	82° 23′	113° 33′	130° $39\frac{1}{2}$′
$20\frac{3}{4}°$	Mean diameter	13° 34′	19° $31\frac{1}{2}$′	31° $30\frac{1}{2}$′	56° 9′	82° 24′	113° $33\frac{1}{2}$′	130° $37\frac{1}{2}$′
	$a-b$ omitting ·00	37954	37953	37950	37951	37952	37954	37953
	Change in $a-b$ per 10′ omitting ·00000	030	037	055	105	140	140	140
	Value of ϕ	85° 37′	83° 40′	79° 53′	72° 15′	64° 45′	57° 11′	53° 58′
	$a-b$ reduced to $22\frac{1}{2}°$ omitting ·00	37944	37946	37943	37944	37945	37947	37949
	$n=310$ for 2nd band	37823	37823	37820	37816	37811	37802	37804

Referring back to Table III. we see that the numbers given there deduced from SARRAU'S theory are uniformly lower. This may be partly due to a wrong value of T, but, as I have said above, I do not think the error in the ratio of the values of T for the two plates exceeded 1 in 10,000. Moreover, SARRAU'S $a-b$ diminishes steadily with ϕ throughout. This is very apparent in the following figures, which are obtained by grouping three or four rings together, allowing as a possible error from the mean the combined effect of $\frac{1}{2}°$ in temperature and $\frac{1}{2}'$ in diameter. In the case of the three small rings, the 2nd, 3rd, and 4th, $\frac{3}{4}'$ is allowed. In the case of the 1st 1′.

From 7 bands	$\phi=90°$ to 53°	$a-b$ lies between	·0037941	and	·0037949
4 rings	$\phi=39°$ to 26°	„	„	·0037928	„	·0037940
3 rings	$\phi=25°$ to 14°	„	„	·0037910	„	·0037934
3 rings	$\phi=13°$ to 8°	„	„	·0037874	„	·0037916

3 rings	$\phi=$ 8° to 6°	$a-b$ lies between	·0037860	and	·0037954
[1 ring	$\phi=$ 4° 24′	„ „	·0037620	„	·0037808

Taking into consideration the probable accuracy of the observations the discrepancy is about equally strongly marked in the case of the first ring and the three rings 13° to 8°, and to a minor degree in the case of some of the others. The observations on Plate 1 are not here included, as we do not know the ratio of the thickness of Plate 1 to that of Plate 3 with sufficient accuracy.*]

Up to this point we have been treating certain constants, f_1 and g_1, which appear in SARRAU's expressions as negligibly small. We will now examine whether by the aid of these constants we can obtain a more satisfactory agreement with observation.

SARRAU's wave-surface is given by

$$(s^2-a^2)(s^2-a^2\cos^2\phi-b^2\sin^2\phi)=\frac{4\pi^2a^2}{\lambda^2}(g_2\cos^2\phi+f_1\sin^2\phi)(g_2\cos^2\phi-g_1\sin^2\phi)$$

Putting $\phi=90°$, we obtain for the equatorial radii α and β.

$$(s^2-a^2)(s^2-b^2)=-\frac{4\pi^2a^2}{\lambda^2}f_1g_1.$$

We might treat this equation to the wave-surface exactly as we have treated the simpler one, and obtain values of $\alpha-\beta$ from the observations on the different values of ϕ. But it may be shown without difficulty that if we equate the values of $a-b$ previously found to

$$(\alpha-\beta)(1+k\cot^2\phi)$$

where

$$k=\frac{8\pi^2a^2}{\lambda^2}\frac{g_2(f_1-g_1)}{(a^2-b^2)^2}$$

we shall attain the same result. In this it is, of course, assumed that f_1 and g_1 are small.

I have found that a suitable value of k is $-$·000033. Using this value we obtain the following figures :—

$\phi=$90° to 53°	$\alpha-\beta$ lies between	·0037941	and	·0037949
39° to 26°	„ „	·0037931	„	·0037943
25° to 14°	„ „	·0037919	„	·0037943
13° to 8°	„ „	·0037914	„	·0037956
8° to 6°	„ „	·0037943	„	·0038037
4° 24′	„ „	·0037830	„	·0038018

* Added May 31, 1886.

Thus the value ·0037943 falls within each pair of limits. The last line of each list gives the results from the first ring obtained with the use of SORET and SARASIN'S value of the rotation. Thus the sudden drop of P_2 for the first ring, which was the most striking eccentricity of the observations on the small rings, is explained at once by the more general form of SARRAU'S theory, without the necessity of assuming an error in the rotational coefficient.

Of course we have had to assign an arbitrary value to k. But the form of the correction, viz., $k \cot^2 \phi$, was fixed by the theory, and was exactly the form that was required, giving, as it does, the correction for $\phi = 4° \, 24'$ some thirty times as great as that for $\phi = 53°$.

We have now compared SARRAU'S theory and the MACCULLAGH group of theories with observation for all our values of ϕ. With regard to the remaining three, CAUCHY has given, so far as I know, no general expression for the wave-surface, while LOMMEL and KETTLER are so widely at variance with the Huyghenian construction that I have not thought it worth while to make a detailed comparison.

I find, however, that the values of $a - b$ deduced on KETTLER'S theory from my observations would differ by about 3 parts in 1000 for $\phi = 60°$ and $\phi = 85°$, whereas according to the Huyghenian construction they agreed within 1 part in 6000.

We conclude, then, that out of the nine theories we have examined the only one which agrees with observation is that of SARRAU, and that to make the agreement as perfect as possible we must choose his constants, so that

$$8\pi^2 a^2 g_2 (g_1 - f_1) = \cdot 000033 (a^2 - b^2)^2 \lambda^2$$

or

$$g_1 - f_1 = 1 \cdot 07 \, g_2.$$

IV.—VALUES OF THE CONSTANTS EMPLOYED.

I have used RUDBERG'S values for a and b, viz.,

$$a = \cdot 647593$$
$$b = \cdot 643799.$$

MASCART gives

$$a = \cdot 647573$$
$$b = \cdot 643756$$

I have taken $\lambda = \cdot 00058920$ mm. This is the mean of ÄNGSTROM'S values for D_1 and D_2.

FIZEAU has given coefficients of the expansion of quartz with temperature along the axis and at right angles to it. He also measured the change of a and b with temperature. These are so small as to be negligible in the present case except for the change of $a-b$. This has been specially investigated by DUFET who finds the change per degree $-\cdot000000415$.

The rotatory constant I have discussed at length above, in connexion with the small rings.

The thickness of Plate 1 was found to be 27·65 mms., of Plate 2 19·977 mms., while the ratio of Plate 3 to Plate 2 was 1·01377.

Plate 2 was cut from the same crystal of quartz as Plate 1.

The rotation of the plane of polarisation produced by either is related to the direction of the light as the rotation to the translation of a right-handed screw. Such is usually called left-handed quartz.

The observations were taken in the Cavendish Laboratory, Cambridge, during the months of March and June, 1885.

V.—PREVIOUS MEASUREMENTS.

The only previous measurements of the retardation in quartz with which I am acquainted are those of JAMIN in 1850 (Ann. de Chim., sér. 3, vol. 30, p. 68), those of HECHT (WIED. Ann., band 20 [1883], p. 426), and my own (Proc. Camb. Phil. Soc., vol. 5 [1883], p. 53).

JAMIN rotated the plate of quartz between two crossed NICOLS as I have done, but he used parallel light, taking the reading when the field of view appeared darkest. Most of the measurements agreed with CAUCHY'S theory within one part in thirty, though there were a few wider discrepancies. This he considered satisfactory agreement. He obtained similar results from another series of experiments in which the axis ratios were also involved.

HECHT'S method was not very dissimilar to that of JAMIN. There is an important oversight in his paper. In estimating the probable error of D he makes the tacit assumption that a certain quantity he calls ϕ is known accurately; while in fact the main part of the error results from the inaccuracy of this ϕ. Correcting this I obtain —using my own notation—for

$$\phi=2^\circ\ 22\cdot4' \qquad D=\cdot1224\ (\pm\cdot0006)$$

$$\phi=7^\circ\ 11\cdot4' \qquad D=\cdot2750\ (\pm\cdot009)$$

whence I deduce for the former $P_2=12\cdot72\pm1\cdot8$, for the latter $P_2=15\cdot82\pm\cdot57$.

The results obtained in my own earlier paper were vitiated, partly by the addition of some corrections founded on an erroneous argument, and partly by the bad faces of the plate examined. The argument was as follows.

The observations were made by bringing the cross wire up to the sensibly dark part of the band, first on one side, then on the other. Then I argued that I knew the position and breadth of the sensibly dark region of the band, but I did not know that by taking the middle of this I should obtain the darkest point. The distance from band to band becomes smaller as we go further from the axis, and I took as an approximation that the darkest point would divide the sensibly dark region in the same ratio as the band itself divides the distance between the two neighbouring bands. Here was the mistake. For the sensibly dark region is that part of the band where the intensity of the light falls below a certain fixed value, and the intensity of the light obviously varies approximately as the square of the small angular distance from the point of total extinction. It might then be expected that this darkest point would lie practically at the middle of the sensibly dark region. This opinion was confirmed by a closer examination, with the aid of Sir GEORGE AIRY'S formula, for the intensity of any point of the field.

By the removal of those corrections the discrepancies I obtained between theory and observation would be reduced one-half, and the remainder may be easily accounted for by the imperfection of the polished faces. I knew that the rings would not be much altered by curvature of the faces; for when the incidence is nearly normal, a slight deviation of the path through the quartz does not materially increase the length of the path, so I contented myself with noticing in the measurements with the callipers that the faces appeared tolerably parallel and plane. Lately, however, I examined the crystal again, and optical tests showed that the curvature of the faces was quite sufficient to explain the discrepancy of some 5′ between theory and observation.